ASYMPTOTIC EQUIPARTITION OF ENERGY FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

BY

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ABSTRACT. Of concern are second order differential equations of the form $(d/dt - if_1(A))(d/dt - if_2(A))u = 0$. Here A is a selfadjoint operator and f_1, f_2 are real-valued Borel functions on the spectrum of A. The Cauchy problem for this equation is governed by a certain one parameter group of unitary operators. This group allows one to define the energy of a solution; this energy depends on the initial data but not on the time t. The energy is broken into two parts, kinetic energy $K(t)$ and potential energy $P(t)$, and conditions on A, f_1, f_2 are given to insure asymptotic equipartition of energy: $\lim_{t \to \pm\infty} K(t) = \lim_{t \to \pm\infty} P(t)$ for all choices of initial data. These results generalize the corresponding results of Goldstein for the abstract wave equation $d^2 u/dt^2 + A^2 u = 0$. (In this case, $f_1(\lambda) \equiv \lambda$, $f_2(\lambda) \equiv -\lambda$.)

1. **Introduction.** Let A be a selfadjoint operator on a complex Hilbert space H. The Cauchy problem

$$(1.1) \qquad u''(t) + A^2 u(t) = 0 \quad (t \in \mathbf{R} = (-\infty, \infty)),$$

$$(1.2) \qquad u(0) = f_1 \in \mathcal{D}(A^2), \quad u'(0) = f_2 \in \mathcal{D}(A)$$

is well posed; here $' = d/dt$ and $\mathcal{D}(A)$ denotes the domain of A. The *energy*

$$E_f = \|u'(t)\|^2 + \|Au(t)\|^2$$

depends on the data $f = (f_1, f_2)$ but not on t. Let

$$K(t) = \|u'(t)\|^2, \quad P(t) = \|Au(t)\|^2$$

denote the *kinetic* and *potential energy* at time t. Goldstein [6] showed that the energy is *asymptotically equipartitioned*:

$$\lim_{t \to \pm\infty} K(t) = \lim_{t \to \pm\infty} P(t) = E_f/2$$

for all choices of initial data f if and only if $\lim_{t \to \pm\infty} \int_{-\infty}^{\infty} e^{it\lambda} d(\|\Pi_\lambda h\|^2) = 0$ for all $h \in H$, where $\{\Pi_\lambda : \lambda \in \mathbf{R}\}$ is the resolution of the identity associated with A, i.e., $A = \int_{-\infty}^{\infty} \lambda \, d\Pi_\lambda$ is the spectral integral representation of A.

Now consider the more general problem

(1.3) $v'(t) = iBv(t)$ $(t \in \mathbf{R})$, $v(0) = g \in \mathcal{D}(B)$

in a complex Hilbert space, where B is selfadjoint. (1.1), (1.2) is a special case of (1.3) (cf. [6]). The solution of (1.3) is given by $v(t) = U(t)g$, where $\{U(t) = \exp(itB): t \in \mathbf{R}\}$ is a one parameter group of unitary operators. Interpret $E_g = \|v(t)\|^2$ to be the energy at time t; E_g depends on the data g but not on t. The *question* which motivated the present paper is: *With which Cauchy problems* (1.3) *can one decompose the energy into different types of energy and prove an asymptotic equipartition of energy theorem?*

This seems to be a very difficult question, even if one specializes the context to symmetric hyperbolic systems (Friedrichs [4]) or higher order hyperbolic equations (Mizohata [8]). Mochizuki [9], [10], generalizing [6], proved an asymptotic equipartition of energy theorem for a special class of higher order hyperbolic equations. In this paper, we generalize [6], [7] in a different way, proving asymptotic equipartition of energy for a class of second order equations of the form

$$u''(t) + iSu'(t) + Tu(t) = 0   (t \in \mathbf{R})$$

where S and T are certain commuting selfadjoint operators.

Other papers dealing with equipartition of energy are Bobisud and Calvert [1], Brodsky [2], Duffin [3], Glassey [5], and Shinbrot [14].

After some preliminaries in §2, the main results are presented in §§3 and 4. §5 is devoted to an example.

We wish to acknowledge some stimulating conversations on equipartition of energy with David Goldstein Costa.

2. **Preliminaries.** Let H be a complex Hilbert space, and let B_1, B_2 be (not necessarily bounded) selfadjoint operators on H. We say that B_1 and B_2 *commute* if and only if $(\lambda iI - B_1)^{-1}$ and $(\mu iI - B_2)^{-1}$ commute for all λ, $\mu \in \mathbf{R}\backslash\{0\}$ (if and only if $\exp(isB_1)$ and $\exp(itB_2)$ commute for all s, $t \in \mathbf{R}$). Also, if B_1, B_2 commute, there is a selfadjoint operator A on H and real-valued Borel functions g_1, g_2 defined on the spectrum of A, $\sigma(A) \subset \mathbf{R}$, such that $B_j = g_j(A)$ for $j = 1, 2$ (cf. [12, p. 358]).

Let B_1, B_2 be commuting selfadjoint operators with $B_j = g_j(A)$ as above. The equation

(2.1) $u'' + iB_1 u' + B_2 u = 0$

can be factored as

(2.2) $(d/dt - if_1(A))(d/dt - if_2(A))u = 0$

where f_1, f_2 are Borel functions on $\sigma(A)$ such that $g_1 = f_1 + f_2$, $g_2 = -f_1 f_2$. There are real-valued solutions f_1, f_2 of these equations as long as $g_1^2 + 4g_2 \geq 0$ on $\sigma(A)$.

Our asymptotic equipartition of energy theorem will be for the Cauchy problem (2.2). As indicated above, a great many of the equations (2.1) can be written in this form. The advantage of working with (2.2) is that one may exploit the techniques developed by Sandefur [13] to deal with the factored equation $\Pi_{j=1}^{m}(d/dt - C_j)u = 0$.

3. The main theorem. Let A be a selfadjoint operator on a complex Hilbert space H. Let f_1, f_2 be real-valued Borel functions on the spectrum $\sigma(A)$ of A. Let $A_j = if_j(A)$ for $j = 1, 2$ and let

$$\mathcal{D}_2 = \bigcap \{\mathcal{D}(A_j A_k): 1 \leq j, k \leq 2\},$$

$$\mathcal{D}_3 = \bigcap \{\mathcal{D}(A_j A_k A_l): 1 \leq j, k, l \leq 2\};$$

$\mathcal{D}_2$ and $\mathcal{D}_3$ are dense in H. Define

$$B = -2^{-1}(A_1 + A_2), \quad C = 2^{-1}(-A_1 + A_2)$$

with domain $\mathcal{D}(B) = \mathcal{D}(C) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$. Note that iB and iC are essentially selfadjoint operators, i.e. their closures are selfadjoint.

We consider the initial value problem

$$(3.1) \qquad (d/dt - A_1)(d/dt - A_2)u(t) = 0 \quad (t \in \mathbf{R}),$$

$$(3.2) \qquad u(0) = \phi_1 \in \mathcal{D}_3, \quad u'(0) = \phi_2 \in \mathcal{D}_2.$$

THEOREM 3.1. *Let A, A_1, A_2 be as above. Then for any $\phi_1 \in \mathcal{D}_3$, $\phi_2 \in \mathcal{D}_2$, the problem (3.1), (3.2) has a unique twice continuously differentiable solution u. Let*

$$K(t) \equiv \|u'(t) + Bu(t)\|, \quad P(t) \equiv \|Cu(t)\|^2, \quad t \in \mathbf{R}.$$

Then

$$E_\phi \equiv K(t) + P(t)$$

depends on ϕ but not on t. Let $\{\Pi_\lambda: \lambda \in \mathbf{R}\}$ be the resolution of the identity associated with A. Then

$$(3.3) \qquad \lim_{t \to \pm\infty} K(t) = \lim_{t \to \pm\infty} P(t) = E_\phi/2$$

for all choices of initial data ϕ as in (3.2) if and only if

$$(3.4) \qquad \lim_{t \to \pm\infty} \int_{-\infty}^{\infty} \exp\{it(f_1(\lambda) - f_2(\lambda))\} d_\lambda(\|\Pi_\lambda x\|^2) = 0$$

for all $x \in H$.

When $f_1(t) \equiv -f_2(t) \equiv t$, (3.1) reduces to (1.1), and Theorem 3.1 contains Goldstein's result [6] as a special case. {Strictly speaking more is demanded of the initial data (cf. (3.2)) than in [6], but there is no loss of generality as long as one works with a dense set of initial data. For example, it suffices to prove the theorem in [6] for initial data $\phi_1, \phi_2 \in \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$; the general case follows by a simple density argument.}

With the aid of the Riemann-Lebesgue lemma it is easy to write down sufficient conditions for (3.4) to hold. For example it is sufficient to suppose A is spectrally absolutely continuous and that for each $x \in H, g_x(h')^{-1} \in L^1(h(\sigma(A)))$ where $g_x(\lambda) = (d/d\lambda)(\|\Pi_\lambda x\|^2)$, and where $h = f_1 - f_2$ is increasing and absolutely continuous on $\sigma(A)$. We omit the simple verification (based on the change of variables $\mu = h(\lambda)$).

PROOF OF THEOREM 3.1. Consider the initial value problem

$$(3.5) \qquad \frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad (t \in \mathbf{R})$$

in $H \oplus H$ with

$$(3.6) \qquad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{D}\left(\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}\right).$$

Since A_j is skew-adjoint it generates a unitary group $\{T_j(t) = \exp(tA_j): t \in \mathbf{R}\}$ for $j = 1, 2$; therefore

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

generates the unitary group

$$T(t) = \left\{\begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} : t \in \mathbf{R}\right\}$$

on $H \oplus H$. Since the operator $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ is bounded,

$$\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

generates a group $\{S(t): t \in \mathbf{R}\}$ given by

$$(3.7) \qquad S(t)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} T_1(t)\psi_1 + \int_0^t T_1(t-s)T_2(s)\psi_2 \, ds \\ T_2(t)\psi_2 \end{pmatrix}$$

according to the Phillips' perturbation theorem [11] and a short calculation. For

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$S(t)\psi$ is the unique twice strongly continuously differentiable solution to (3.5) and (3.6). (One can, of course, easily directly check that for $S(t)\psi$ defined by (3.7),

$$(d/dt)S(t)\psi = \begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix} S(t)\psi,$$

$S(0)\psi = \psi$.)

If we choose $\psi_1 = \phi_1$, $\psi_2 = \phi_2 - A_1\phi_1$ and set

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = S(t)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad t \in \mathbf{R},$$

it follows that u_1 is a solution of (3.1), (3.2). Also by (3.2) and the definition of $\mathcal{D}_2$ and $\mathcal{D}_3$, we have

$$\psi_1, \psi_2 \in \mathcal{D}\left(\begin{pmatrix} A_1 & I \\ 0 & A_2 \end{pmatrix}^2\right)$$

and so u_1 and $u_2 = u_1' - A_1 u_1$ are twice strongly continuously differentiable.

If v is any twice strongly continuously differentiable solution to (3.1), (3.2), then

$$\begin{pmatrix} v \\ v' - A_1 v \end{pmatrix}$$

is a solution to (3.5), (3.6). But by uniqueness for (3.5), (3.6), it follows that $v = u_1$, and so the solution to (3.1), (3.4) is unique.

Note that

$$(3.8) \quad \|u_1'(t) - A_1 u(t)\|^2 = \|u_2(t)\|^2 = \|T_2(t)(\phi_2 - A_1\phi_1)\|^2 = \|\phi_2 - A_1\phi_1\|^2$$

since $T_2(t)$ is unitary. We have the companion equality

$$(3.9) \qquad \|u_1'(t) - A_2 u_1(t)\|^2 = \|\phi_2 - A_2\phi_1\|^2$$

for all $t \in \mathbf{R}$. That we can interchange the subscripts 1 and 2 in ϕ and A follows from that fact that $(d/dt - A_1)(d/dt - A_2)u_1 = 0$ if and only if $(d/dt - A_2) \cdot (d/dt - A_1)u_1 = 0$ by the closedness and commutativity of A_1 and A_2; thus the proof of (3.8) also yields (3.9). From now on let u $(=u_1)$ denote the unique

solution of (3.1), (3.2). By (3.8), (3.9) and the parallelogram law,

$$
\begin{aligned}
K(t) + P(t) &= \|u'(t) + Bu(t)\|^2 + \|Cu(t)\|^2 \\
&= 2^{-1}\{ \|u'(t) + Bu(t) - Cu(t)\|^2 + \|u'(t) + Bu(t) + Cu(t)\|^2 \} \\
&= 2^{-1}\{ \|u'(t) - A_2 u(t)\|^2 + \|u'(t) - A_1 u(t)\|^2 \} \\
&= 2^{-1}\{ \|\phi_2 - A_2\phi_1\|^2 + \|\phi_2 - A_1\phi_1\|^2 \} \\
&= \|\phi_2 + B\phi_1\|^2 + \|C\phi_1\|^2 = E_\phi
\end{aligned}
$$

for all $t \in \mathbf{R}$. Thus the first two assertions of the theorem are proved. Next, using (3.8), (3.9) again,

$$
\begin{aligned}
P(t) = \|Cu(t)\|^2 &= 4^{-1} \|(u'(t) - A_1 u(t)) - (u'(t) - A_2 u(t))\|^2 \\
&= 4^{-1}\{ \|u'(t) - A_1 u(t)\|^2 + \|u'(t) - A_2 u(t)\|^2 \\
&\qquad - 2 \, \mathrm{Re}\langle T_1(t)(\phi_2 - A_2\phi_1), T_2(t)(\phi_2 - A_1\phi_1)\rangle \} \\
&= 4^{-1}\{ \|\phi_2 - A_1\phi_1\|^2 + \|\phi_2 - A_2\phi_1\|^2 \\
&\qquad - 2 \, \mathrm{Re}\langle T_2(-t)T_1(t)(\phi_2 - A_2\phi_1), \phi_2 - A_1\phi_1\rangle \} \\
&= 2^{-1} E_\phi - \alpha(t, \phi)
\end{aligned}
$$

(3.10)

where

$$
(3.11) \qquad \alpha(t, \phi) = 2^{-1} \, \mathrm{Re}\langle T_2(-t)T_1(t)(\phi_2 - A_2\phi_1), \phi_2 - A_1\phi_1\rangle.
$$

Thus $\lim_{t \to \pm\infty} P(t) = E_\phi/2$ for all ϕ as in (3.2) if and only if $\lim_{t \to \pm\infty} \alpha(t, \phi) = 0$ for all such ϕ. Taking $\phi_1 = 0$, we see that (3.3) implies

$$
(3.12) \qquad \lim_{t \to \pm\infty} \mathrm{Re}\langle T_2(-t)T_1(t)x, x\rangle = 0
$$

for all $x \in \mathcal{D}_2$, whence for all $x \in H$. But (3.12) implies

$$
(3.13) \qquad \lim_{t \to \pm\infty} \langle T_2(-t)T_1(t)x, y\rangle = 0
$$

for all $x, y \in H$ by polarization. Thus (3.3) implies

$$
(3.14) \qquad \lim_{t \to \pm\infty} \alpha(t, \phi) = 0;
$$

simply take $x = \phi_2 - A_2\phi_1, y = \phi_2 - A_1\phi_1$ in (3.13). Conversely, (3.12) implies (3.13) implies (3.14) implies (3.3); thus (3.3) and (3.12) are equivalent.

According to the spectral theorem and the associated operational calculus,

$$
\langle T_2(-t)T_1(t)x, x\rangle = \int_{-\infty}^{\infty} \exp\{ it(f_1(\lambda) - f_2(\lambda))\} \, d_\lambda\langle \Pi_\lambda x, x\rangle,
$$

whence (3.12) is equivalent to (3.4) and the theorem follows. □

4. Asymptotic equipartition in the Cesàro sense.

THEOREM 4.1. *Let A, A_1, A_2 be as in Theorem 3.1. Then*

$$(4.1) \qquad \lim_{\tau \to \pm \infty} \tau^{-1} \int_0^\tau K(t)\, dt = \lim_{\tau \to \pm \infty} \tau^{-1} \int_0^\tau P(t)\, dt = E_\phi/2$$

for all initial data $\phi = (\phi_1, \phi_2) \in \mathcal{D}_3 \times \mathcal{D}_2$ if and only if

$$(4.2) \qquad \lim_{\tau \to \pm \infty} \tau^{-1} \int_0^\tau \alpha(t, \phi)\, dt = 0$$

for all such ϕ where α is defined by (3.11).

In particular, (4.1) holds if 0 is not an eigenvalue of A (i.e. A is one-to-one) and for each $\delta > 0$,

$$(4.3) \qquad \inf\{ |f_1(t) - f_2(t)| : t \in \sigma(A),\ |t| \geq \delta \} > 0.$$

This generalizes a theorem in [7] in the same way that Theorem 3.1 generalized [6].

PROOF OF THEOREM 4.1. According to (3.10), $P(t) = 2^{-1} E_\phi - \alpha(t, \phi)$ for all initial data ϕ. The equivalence of (4.1) and (4.2) follows easily from this and from $E_\phi = K(t) + P(t)$. By Theorem 3.1 it also follows that (4.2) holds if and only if

$$\lim_{\tau \to \pm \infty} \tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)x, x \rangle \, dt = 0$$

for all $x \in H$. If the hypotheses of the second paragraph of the theorem hold, we have

$$\tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)\lambda, x \rangle \, dt = J_1 + J_2$$

$$\equiv \tau^{-1} \int_0^\tau \int_{|\lambda| \leq \delta} \exp\{it(f_1(\lambda) - f_2(\lambda))\} \, d(\|\Pi_\lambda x\|^2)$$

$$+ \tau^{-1} \int_0^\tau \int_{|\lambda| > \delta} \exp\{it(f_1(\lambda) - f_2(\lambda))\} \, d(\|\Pi_\lambda x\|^2);$$

$$|J_1| \leq \tau^{-1} \int_0^\tau \int_{|\lambda| \leq \delta} d(\|\Pi_\lambda x\|^2) \leq \|\Pi_{\delta+} x - \Pi_{\delta-} x\|^2.$$

Given $\epsilon > 0$, choose and fix $\delta > 0$ such that $|J_1| < \epsilon/2$; this can be done since A is one-to-one. Next, if $b > 0$ represents the left-hand side of (4.3) and if $h = f_1 - f_2$,

$$|J_2| = \left| \int_{|\lambda| > \delta} \tau^{-1} \int_0^\tau \exp\{ith(\lambda)\} \, dt \, d(\|\Pi_\lambda x\|^2) \right|$$

$$= \left| \int_{|\lambda| > \delta} [\exp\{i\tau h(\lambda)\} - 1] [i\tau h(\lambda)]^{-1} \, d(\|\Pi_\lambda x\|^2) \right|$$

$$\leq 2\|x\|^2 |\tau b|^{-1} < \epsilon/2.$$

if $|\tau| > 4\|x\|^2/\epsilon b \equiv \tau(\epsilon, x)$. Consequently for each $x \in H$ and $\epsilon > 0$, $|\tau| > \tau(\epsilon, x)$ implies

$$\left| \tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)x, x \rangle \, dt \right| < \epsilon,$$

and so (4.1) holds. $\square$

If $Ah = 0$ for some $h \neq 0$ and if $f_1(0) = f_2(0)$, then $T_1(t)h = T_2(t)h = h$ for all real t, whence

$$\tau^{-1} \int_0^\tau \langle T_2(-t)T_1(t)h, h \rangle \, dt = \|h\|^2$$

does not tend to zero as $\tau \to \pm \infty$, hence (4.2) and (4.1) fail to hold in this case.

5. **An example.** The maximal operator A for id/dx on the complex Hilbert space $H = L^2(\mathbf{R})$ is selfadjoint. Consider the initial value problem for

(5.1) $u''(t) + P(A)u'(t) + Q(A)u(t) = 0 \qquad (t \in \mathbf{R})$

where

$$P(\xi) = \sum_{j=0}^n a_j \xi^j, \qquad Q(\xi) = \sum_{j=0}^m b_j \xi^j/4$$

for $\xi \in \mathbf{R}$, where a_j, $b_j \in \mathbf{C}$, $a_n \neq 0$, $b_m \neq 0$. (Condition (3.2) in this case becomes $u(0) \in \mathcal{D}(A^{3N})$, $u'(0) \in \mathcal{D}(A^{2N})$ where $N = \max(n, m)$.) (5.1) can be rewritten in the form

(5.2) $[d/dt - 2^{-1}(-P(A) + R(A))] [d/dt - 2^{-1}(-P(A) - R(A))] u = 0$

where

$$R(\xi) = \{P(\xi)^2 - 4Q(\xi)\}^{1/2}, \qquad \xi \in \mathbf{R}.$$

Thus Theorem 3.1 (and Theorem 4.1) can be applied if $\mathrm{Re}(-P(\xi) \pm R(\xi)) = 0$ for each $\xi \in \mathbf{R}$. This is equivalent to:

(i) For $j = 0, 1, \ldots, n$, $a_j = ic_j$ where $c_j \in \mathbf{R}$;

(ii) $-(\Sigma_{j=0}^n c_j \xi^j)^2 - \Sigma_{j=0}^m b_j \xi^j$ is real and nonpositive for all real ξ.

In particular, each b_j must be real. We now suppose $c_j = -ia_j$ and b_k are real for each j, k.

Case I. Suppose $2n > m$. The left side of (ii) is bounded above. Consequently (ii) automatically holds if we increase b_0 enough.

Case II. Suppose $2n = m$. If $-c_n^2 - b_{2n} < 0$, the left side of (ii) is bounded above, so (ii) holds if we increase b_0 enough.

Case III. Suppose $2n < m$. In order for the left-hand side of (ii) to be bounded above, m must be even. If this is so and if $b_m > 0$, then the conclusion of Cases I and II hold in this case as well. Thus we assume one of the following conditions:

(α) $2n > m$, or

(β) $2n = m$, $c_n^2 + b_{2n} > 0$, or

(γ) $2n < m$, m is even, $b_m > 0$,

and if necessary we enlarge b_0 so that (i) and (ii) hold.

Let $\{\Pi_\lambda : \lambda \in \mathbf{R}\}$ be the resolution of the identity associated with A and let $\{\Gamma_\mu : \mu \geqslant 0\}$ be the resolution of the identity associated with the positive operator $S(A) = \{4Q(A) - P(A)^2\}^{1/2}$. Both A and $S(A)$ are spectrally absolutely continuous. Let

$$\mathcal{D} = \{x \in H : \Pi_\lambda x - \Pi_{-\lambda} x = \lambda \text{ for some } \lambda > 0\}.$$

$\mathcal{D}$ is dense in H. To simplify the calculation we suppose that $S(\xi) = \{4Q(\xi) - P(\xi)^2\}^{1/2}$ is monotone decreasing on $(-\infty, L)$ and monotone increasing on (L, ∞). (Without this assumption the following computations become more complicated, but the results remain valid.) For $x \in \mathcal{D}$, choose M so that $(\Pi_M - \Pi_{-M})x = x$. Then

$$\int_{-\infty}^{\infty} \exp(it(f_1(\lambda) - f_2(\lambda)))\, d(\|\Pi_\lambda x\|^2) = \int_{-\infty}^{\infty} e^{itS(\lambda)}\, d(\|\Pi_\lambda x\|^2)$$

$$= \int_{-M}^{L} e^{itS(\lambda)}\, d(\|\Pi_\lambda(\Pi_L x)\|^2) + \int_{L}^{M} e^{itS(\lambda)}\, d(\|\Pi_\lambda(x - \Pi_L x)\|^2)$$

$$= \int_{S(L)}^{S(-M)} e^{-it\mu}\, \frac{d}{d\lambda}(\|\Pi_\lambda(\Pi_L x)\|^2)\frac{d}{d\mu}(S^{-1}(\mu))\, d\mu$$

$$+ \int_{S(L)}^{S(M)} e^{it\mu}\, \frac{d}{d\lambda}(\|\Pi_\lambda(x - \Pi_L x)\|^2)\frac{d}{d\mu}(S^{-1}(\mu))\, d\mu$$

$$\longrightarrow 0 \quad \text{as } t \longrightarrow \pm\infty$$

by the Riemann-Lebesgue lemma. Since $\mathcal{D}$ is dense, a simple density argument shows that (3.4) holds for all $x \in H$. Thus Theorem 3.1 gives asymptotic equipartition of energy for the equation

$$(\partial^2/\partial t^2 + P(i\partial/\partial x)(\partial/\partial t) + Q(i\partial/\partial x))u(t, x) = 0$$

where

$$P\left(i\frac{\partial}{\partial x}\right) = \sum_{j=0}^{n} i^{j+1} c_j \left(\frac{\partial}{\partial x}\right)^j, \quad Q\left(i\frac{\partial}{\partial x}\right) = 4^{-1} \cdot \sum_{j=0}^{m} i^j b_j \left(\frac{\partial}{\partial x}\right)^j,$$

c_j, $b_k \in \mathbf{R}$, $c_n \neq 0$, $b_m \neq 0$, and (α) or (β) or (γ) hold (and b_0 is large enough). □

CONCLUDING REMARK. It would be of interest to find an equipartition of energy theorem subsuming both the result of this paper and the result of Mochizuki [9].

ADDED IN PROOF. David G. Costa has informed us that he has obtained a result similar to Theorem 3.1.

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